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# EQUALIZATION STUDIES FOR THE PAM DIGITAL MODEM

A. E. Post

MARCH 1970

Prepared for

## DIRECTORATE OF PLANNING AND TECHNOLOGY
### ELECTRONIC SYSTEMS DIVISION
### AIR FORCE SYSTEMS COMMAND
### UNITED STATES AIR FORCE
L. G. Hanscom Field, Bedford, Massachusetts

Project 700G

Prepared by

THE MITRE CORPORATION
Bedford, Massachusetts

Contract F19(628)-68-C-0365

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## FOREWORD

This report covers work accomplished for a Digital Communications Project (700G) by The MITRE Corporation, Bedford, Mass., under Contract No. F19(628)-68-C-0365. The contract sponsor was the Electronic Systems Division of the Air Force Systems Command.

## REVIEW AND APPROVAL

This technical report has been reviewed and is approved.

JOSEPH N. ALLRED, Captain, USAF
Staff, Development Engineering Division
Directorate of Planning and Technology

## ABSTRACT

This paper is a revision of WP-2888. It reviews the present state-of-the-art techniques of narrowband (4 KHz) wireline equalization for PAM signals. Two basic minimization techniques are discussed. Related derivations and proofs are included. In addition to linear distortion, the problems due to phase jitter are mentioned as areas of possible future research.

# TABLE OF CONTENTS

## LIST OF ILLUSTRATIONS

SECTION I

1.0 INTRODUCTION

In the relatively noisefree (S/N > 26 db) environment of wire-
line data communications, linear distortion in the transmission
system is a major concern when trying to achieve high information
throughput. Data pulses sent through such a transmission line will
tend to overlap upon their neighbors causing intersymbol interference.
To minimize this distortion, equalizers consisting of tapped delay
lines with adjustable tap gains have been considered [1]-[11].
Another serious contribution to signal distortion is the presence
of incidental phase modulation caused by the transmission system's
carrier multiplex equipment. The purpose of this report is to
discuss the basic concepts and techniques of signal equalization
for multilevel PAM signals in order to develop and quantitatively
evaluate proposed signal processing techniques (primarily digital)
that are intended to cope with the above-mentioned signal distortion
effects.

The discussion given here is concentrated on time domain digital
equalization techniques which make use of tapped delay lines with
appropriately weighted tap gains. Presently, equalizers are designed
so that tap gains are adjusted using adaptive algorithms that operate
while data is being transmitted. These methods have been shown to
work better than preset algorithms[8] and are relatively easy to

1

implement. Thus, the problem of linear distortion can be handled without much difficulty.

The problem of compensating for phase jitter (modulation), on the other hand, has not yet been solved. Phase jitter causes one to sample at non-optimum times when distortion is enough to greatly increase the error rate. The nature of phase jitter is examined in terms of time and frequency domain response and interaction with the modulated data signal. Techniques being used in newly developed modems are described and their limitations are considered. Thus, further consideration must be given to techniques for controlling sampling time[10] and "tracking out" phase jitter.

## SECTION II

## 2.0 OPTIMAL TRANSVERSAL EQUALIZERS

Figure 1 illustrates a transversal equalizer. Its output, $y(t)$, is a weighted sum of delayed (or advanced) values of input $x(t)$:

$$y(t) = \sum_{j=-n}^{n} c_j x(t-jT) \quad , \tag{1}$$

where $x(t)$ is the delayed value of the input signal that appears at the center tap, $T$ is the amount of delay between each tap, and $\{c_j\}$ is the set of $N = 2n + 1$ tap gains. Because our concern is with the sampling instant, one can rewrite equation (1):

$$y_k = \sum_{j=-n}^{n} c_j x_{k-j} \quad , \tag{2}$$

where $y_k = y(kT)$ and $x_{k-j} = x(kT-jT)$. Assume that $x(t)$ is the result of sending a single pulse of unit amplitude through the system. Define $h(t)$ as the output, $y(t)$, that consequently comes from the equalizer. Ideally, we want the sampled values of $h(t)$ to be given by:

$$h_k = \left\{ \begin{array}{lll} 1 & , & k=0 \quad ; \\ 0 & , & k \neq 0 \quad . \end{array} \right. \tag{3}$$

3
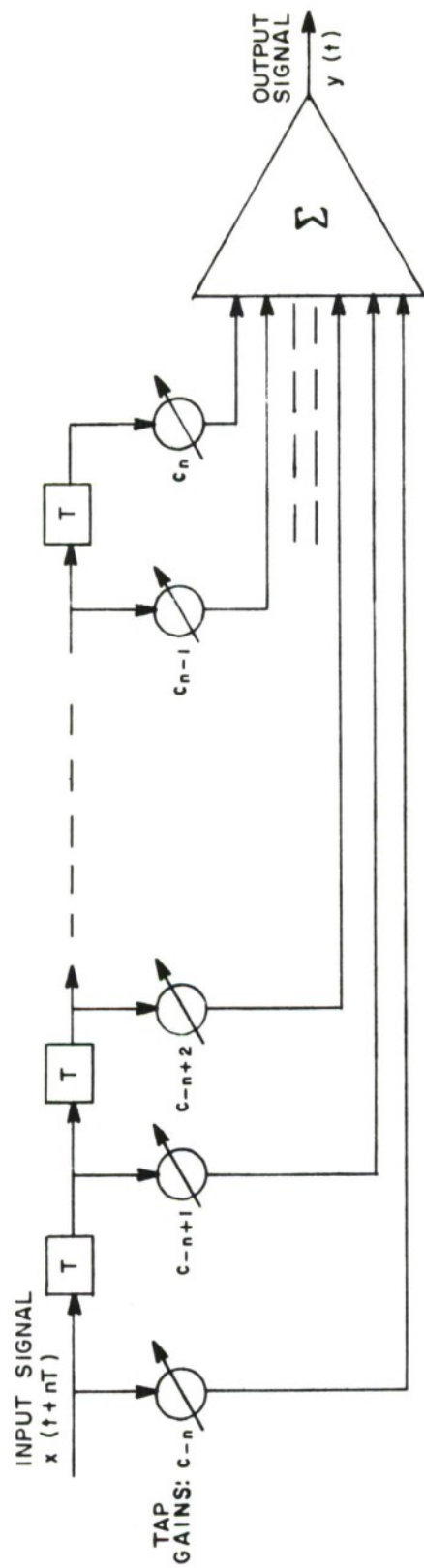
Figure 1 TRANSVERSAL EQUALIZER

However, having a finite number, N, of taps permits a suboptimal solution, satisfying equation (3) for at most N values of k. Lucky has shown[7] that if the eye pattern for x(t) is open, the optimal solution for N taps is given by:

$$u_k = \sum_{j=-n}^{n} c_j x_{k-j} \quad , \quad |k| \le n \quad ; \tag{4}$$

where

$$u_k = \begin{cases} 1 & , \quad k=0 \quad ; \\ 0 & \quad k \ne 0 \quad . \end{cases} \tag{5}$$

In matric notation:

$$\underline{u} = \underline{X} \, \underline{c} \tag{6}$$

where $\underline{u}$ and $\underline{c}$ are N-vectors and $\underline{X}$ is an N x N matrix whose components are samples of the response of the system to a unit pulse. If the eye of x(t) is initially open, then equation (6) can be solved:

$$\underline{c} = \underline{X}^{-1} \, \underline{u}$$

5

where $\underline{X}^{-1}$ is the inverse of matrix $\underline{X}$. If the eye is not open, $\underline{X}^{-1}$ may not necessarily exist or, if it does, equation (7) may not yield the optimal solution. Should the former be true, one might try to minimize the square of the distance between $\underline{X}\ \underline{c}$ and $\underline{u}$:

$$L \overset{\Delta}{=} (\underline{X}\ \underline{c} - \underline{u})^T\ (\underline{X}\ \underline{c} - \underline{u}) \quad , \tag{8}$$

where superscript "T" indicates a matrix (or vector) transpose. L is minimized by setting

$$\frac{\partial L}{\partial \underline{c}} = \underline{0} \tag{9}$$

to get

$$\underline{c} = (\underline{X}^T\underline{X})^{-1}\ \underline{X}^T\underline{u} \tag{10}$$

Note that equation (10) reduces to equation (7) when $\underline{X}$ is non-singular (i.e., $\underline{X}^{-1}$ exists).

6

## SECTION III

3.0 ITERATIVE PRESET ALGORITHM[7]

An iterative algorithm is necessary in order to average out any effects of noise on the $x(t)$ measurements. An iterative algorithm similar to Lucky's[7] requires the transmission of a series of test pulses that are spaced relatively far apart so as not to overlap. Define error vector $\underline{q}$:

$$\underline{q} \triangleq \underline{h} - \underline{u} \quad , \tag{11a}$$

or

$$q_k \triangleq \left\{ \begin{array}{ll} h_k & , \quad k \neq 0 \\ h_0 - 1 & , \quad k = 0 \end{array} \right. \quad . \tag{11b}$$

After each test pulse, the tap gains are changed as follows:

$$\Delta c_j = - \epsilon \, \text{sgn} \, q_j \tag{12}$$

where $\epsilon$ is the smallest increment one can make for a tap gain. Convergence of the algorithm given by equation (12) is proven in Appendix A.

7

## 3.1 Adaptive Version of Preset Algorithm[8]

The adaptive version of the preset algorithm is one that makes an estimate of each $q_j$ during a period of data transmission and then updates the taps at the end of that averaging period according to equation (12). Appendix B shows that the estimates are made as follows:

$$\hat{q}_k = \frac{1}{KS} \sum_{k=1}^{K} e_k a_{k-j} \quad , \tag{13}$$

where K is the number of samples used to reach a decision, S is the average signal power, $a_k$ is the amplitude of the $k^{th}$ message, and $e_k = y_k - a_k$ where $y_k$ is the $k^{th}$ output sample. Note that if the eye is initially closed, errors will be made when deciding upon the $a_k$'s. Consequently, the $\hat{q}_j$'s may well be incorrect and convergence may not necessarily occur.

## 3.2 Digitized Adaptive Equalizer[8]

Lucky simplified the previous algorithm considerably by digitizing and allowing its rate of tap gain update to be variable. Rather than correlating analog data, as in euqation (13), the digital algorithm employs, for the $i^{th}$ tap, an up-down counter that counts up when sgn $e_k$ = sgn $a_{k-i}$ and down when sgn $e_k \neq$ sgn $a_{k-i}$. When a counter reaches one of its two thresholds, it will cause its tap gain to either increase or decrease by $\epsilon$, depending upon whether the lower or higher threshold is crossed; upon reaching a threshold,

8

it is reset to the center of its range.  The probability of correct
update (estimate of $q_i$) as well as speed of convergence is determined
by the thresholds of the counters.

Lucky has shown[8] that for a given amount of accuracy, $\epsilon$, this
digitized equalizer converges faster than the one mentioned in the
previous section.  This occurs in spite of the use of the signum
function to discard much information.  Use of the non-linear
sequential test in the digital version causes taps to update much
more often than every K samples when equalization is poor.  Also,
when the equalization in the digital algorithm is fairly accurate,
the taps are not forced to change as often as with the other, thus
providing greater accuracy.

# SECTION IV

## 4.0 FULL EQUALIZER[5]

Equation (13) helps illustrate that to maximize the eye opening, one must reduce the cross correlation between the transmitted message and the quantization error in the equalized signal. Appendix C shows that to minimize the mean-squared value of the quantization error mentioned above, one must reduce the cross-correlation between the error and the unequalized input signal. Thus,

$$\Delta c_k^j = - \epsilon \, x_{j-k} \, e_j \qquad (14)$$

The full mean-squared error equalizer is illustrated in Figure 2. In Appendix D, the mean-squared error algorithm is shown to always converge when $\epsilon$ is chosen properly.

## 4.1 "Hybrid" Algorithm[6]

The major drawback of the MSE equalizer illustrated in Figure 2 is the necessity of two multiplications and an accumulation for each tap during a single baud. To reduce this complexity, the Hybrid algorithm calls for truncation of the error signal to $\pm \epsilon$. Thus, the algorithm for tap update would be:

$$\Delta c_k^j = - \epsilon \, x_{j-k} \, SGN \, (e_j) \qquad (15)$$

10
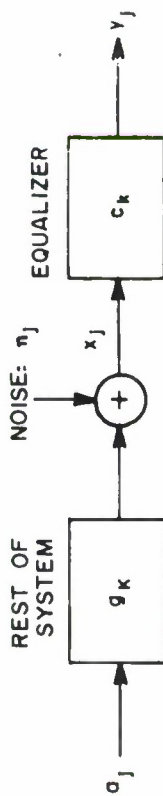
Figure 2 SAMPLED DATA MODEL OF SYSTEM

Hirsch and Wolf[6] have results indicating that this algorithm does converge, but not as fast as the full MSE--logical because information is discarded in taking the sign of the quantization error.

## 4.2 "Modified Zero-Forcing" Algorithm[6]

This is the algorithm preferred by Hirsch and Wolf[6] because it is as easy to implement as the digitized adaptive zero-forcing algorithm of Section 3.2, yet, seems to always converge (although not quite as fast as the two previous MSE algorithms). The simplification is the truncation of the tap signal that is to be correlated with the truncated error signal:

$$\Delta c_k^j = - \epsilon \; SGN \; (x_{j-k}) \; SGN \; (e_j) \tag{16}$$

Thus, an up-down counter can be used here to control the tap gain settings.

12

SECTION V

5.0 COMPUTER SIMULATION

Simulation of a realistic data transmission system having a mean squared error equalizer (Section 4.0) were run on the IBM 360/50 in the Fortran IV language. Figure 3 is a model of that system (see MTR-893). Its components were simulated as follows:

(1) The binary message, $a_k = \pm 1$, is a random binary sequence without any self correlation ($a_k$ can also be taken from a multilevel signal set containing $2^n$ levels where n is the number of bits in the message). This constraint is necessary for the equalizer to adapt for all frequencies.

(2) Wave shaping is accomplished by using the time domain representation of the waveform, in this case, a signal whose Fourier transform is a raised cosine pulse--a periodic signal with no intersymbol interference at its sampling time. Time truncation is also performed--a practical necessity.

(3) After applying the fast Fourier transform, the signal is lowpassed (truncated in the frequency domain) to prevent aliasing when modulating.

(4) Modulation is accomplished by merely translating in the frequency domain.

13
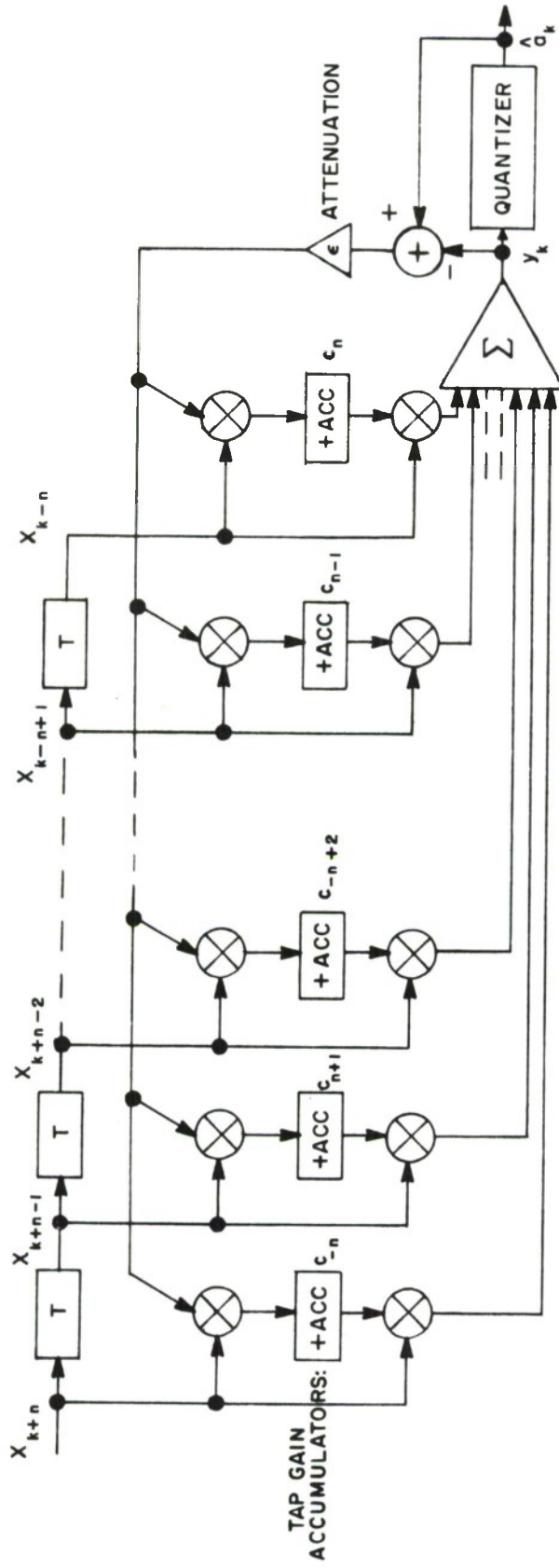
Figure 3 MEAN–SQUARED ERROR EQUALIZER

(5) Because of limited channel bandwidth, only the lower side-band is to be sent. This is accomplished using a filter with symmetric rolloff about the carrier, in this case: raised cosine rolloff. To partially compensate for the line, predistortion was introduced.

(6) The channel model was limited to linear distortion. This, again, was accomplished in the frequency domain.

(7) Coherent demodulation, a necessity for VSB, was accomplished by using the phase of $f_c$, the carrier frequency, and then translating down in frequency after adjusting for phase.

(8) Noise was added in the time domain after taking the inverse fast Fourier transform of the demodulated signal.

(9) The equalizer used has 21 taps; $\epsilon$ varied between $2^{-9}$ and $2^{-7}$, where $\epsilon$ is the gain in the tap gain update loop.

After the simulations were run, it was decided that one system configuration improvement would consist of using a matched filter at the receiver. Thus, the waveshaper would send a signal whose transform is $\sqrt{H(f)}$ and the receiver's filter would be the same--allowable because $H(f)$ is pure real. One advantage of having a matched filter is an improved signal to noise ratio.

15

Unfortunately, the running time for the above simulation is excessive. One thousand iterations (.3125 sec real time) ran for approximately forty-five minutes. Convergence had not yet been achieved. Thus, one is compelled to re-evaluate the trade-off between computer simulation and hardware experiments. One obvious approach is to fabricate a hardware model of the complete modem. However, this would be expensive and time consuming. A more flexible and cost effective approach is the hybrid combination of computer simulation to generate the equivalent receiver signal waveform as it would exist just prior to equalization and then convert this signal (by appropriate D/A conversion) to an analogue signal input to a breadboard model of the proposed equalizer. The necessary equipment to implement such a process is already available or being procured.

6.0  OPTIMIZATION OF SAMPLING TIME

Transversal equalizers, by their very nature, improve the probability of error-free detection in a highly localized region about the sampling instant.  Phase jitter will severely degrade the performance provided through equalization by causing the receiver to, in effect, sample at times other than the proper sampling time. Appendix E shows how bad the degradation can be.  Consequently, it is important to know the amount of phase jitter at any given instant in order to sample at the correct time.  Accurately tracking the phase jitter, however, is a major problem.

In order to achieve higher data rates, it is necessary to use as much bandwidth as possible.  For PAM-VSB (supressed carrier) modulation schemes, a carrier somewhere in the range 2 - 3 KHz will almost always be in the midst of data.  Consequently, accurate tracking of the carrier phase is not possible here because of interference caused by the signal.  (One must track with a bandwidth of approximately 600 Hz.)  Some modulation schemes shape the transmitted signal by inserting a notch at the carrier to allow the addition of a pure carrier tone.  Because of the necessity of wide tracking bandwidth, this, too, has proven to be unsuccessful.

One untried possibility utilizes the tracking of a tone other than the carrier. Others might use non-linear estimation techniques. Further improvement might entail choice of a sampling time where the eye pattern resulting after equalization is wide for a greater range of time. Thus, further investigation is required.

## SECTION VII

7.0 CONCLUSION

The bulk of this work has been devoted to the presentation of the current state-of-the-art of equalization. For a data quality telephone channel, the algorithm that is superior in terms of both cost and effectiveness has not yet been determined. In addition, in order to achieve sufficiently high data rates, it will be necessary to compensate for phase jitter. Such compensation may be integrated into an equalization algorithm, both of which require further study.

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## APPENDIX A

### Convergence of the Iterative Preset Algorithm

The algorithm for change of tap gain, $c_j$, is given by equation (12):

$$\Delta c_j = - \epsilon \, SGN \, q_j \qquad (A-1)$$

or

$$\Delta \underline{c} = - \epsilon \, SGN \, \underline{q} \qquad (A-2)$$

From equations (2) and (11a):

$$\underline{q} = \underline{X} \, \underline{c} - \underline{u} \qquad (A-3)$$

Thus,
$$\underline{q}^{(n+1)} = \underline{X} \, \underline{c}^{(n+1)} - \underline{u} \qquad (A-4)$$

$$\underline{q}^{(n+1)} = \underline{X} \, (\underline{c}^n + \Delta \underline{c}) - \underline{u} \qquad (A-5)$$

From equation (A-2):

$$\underline{q}^{(n+1)} = \underline{X} \, \underline{c}^{(n)} - \epsilon \, \underline{X} \, SGN \, \underline{q}^{(n)} - \underline{u} \qquad (A-6)$$

From equation (A-3):

$$\underline{q}^{(n+1)} = \underline{q}^{(n)} - \epsilon \, \underline{X} \, SGN \, \underline{q}^{(n)} \qquad (A-7)$$

23

<div align="center">or</div>

$$q_k^{(n+1)} = q_k^{(n)} - \epsilon \sum_{j=-n}^{n} (SGN \ q_j^{(n)}) \ x_{k-j} \qquad (A-8)$$

$$q_k^{(n+1)} = q_k^{(n)} - \epsilon \ x_0 \ SGN \ q_k^{(n)} - \epsilon \sum_{\substack{j=-n \\ j \neq k}}^{n} (SGN \ q_j^{(n)}) \ x_{k-j} \qquad (A-9)$$

The normalized distortion of $x(t)$ before equalization is defined as

$$D \overset{\Delta}{=} \frac{1}{x_0} \sum_{\substack{j=-\infty \\ j \neq 0}}^{\infty} |x_j| \quad , \qquad (A-10)$$

where $x_o$ is assumed to be positive. Using Schwarz's Inequality on equation (A-9):

$$|q_k^{(n+1)}| \leq |q_k^{(n)} - \epsilon \ x_0 \ SGN \ q_k^{(n)}| + \epsilon \left| \sum_{\substack{j=-n \\ j \neq k}}^{n} (SGN \ q_j^{(n)}) \ x_{k-j} \right| \qquad (A-11)$$

Note the following:

$$|q_k^{(n)} - \epsilon \ x_0 \ SGN \ q_k^{(n)}| = ||q_k^{(n)}| - \epsilon \ x_0| \qquad (A-12)$$

<div align="center">24</div>

$$\left| \sum_{\substack{j=-n \\ j \neq k}}^{n} (\text{SGN } q_j^{(n)}) x_{k-j} \right| \leq \sum_{\substack{j=-n \\ j \neq k}}^{n} \left| (\text{SGN } q_j^{(n)} x_{k-j} \right| \leq \sum_{\substack{j=-n \\ j \neq k}}^{n} |x_{k-j}| \leq \sum_{\substack{j=-\infty \\ j \neq 0}}^{\infty} |x_j|$$

(A-13)

Thus, using equations (A-10), (A-11), (A-12) and (A-13):

$$\left| q_k^{(n+1)} \right| \leq \left| |q_k^{(n)}| - \epsilon \, x_0 \right| + \epsilon \, x_0 \, D$$

(A-14)

If the eye is assumed to be initially open, $D < 1$; hence

$$\left| q_k^{(n+1)} \right| < \left| |q_k^{(n)}| - \epsilon \, x_0 \right| + \epsilon \, x_0$$

(A-15)

If $\left| q_k^{(n)} \right| > \epsilon x_0$, equation (A-15) becomes

$$\left| q_k^{(n+1)} \right| < \left| q_k^{(n)} \right|$$

(A-16)

If $\left| q_k^{(n)} \right| < \epsilon x_0$, equation (A-15) becomes

$$\left| q_k^{(n+1)} \right| < 2\epsilon x_0 - \left| q_k \right|$$

(A-17)

$$< 2\epsilon x_0$$

(A-18)

25

Thus, when the eye is initially open, each $q_k$ will converge to within $2\epsilon x_0$ of zero.

# APPENDIX B

## Estimation of $q_j$

When transmitting data, the output is due to past and future input pulses as well as noise:

$$y_k = \sum_{j=-\infty}^{\infty} a_j h_{k-j} + n_k = \sum_{j=-\infty}^{\infty} a_j q_{k-j} + a_k + n_k \qquad \text{(B-1)}$$

where the $a_j$'s are the input amplitudes and the $n_k$'s are noise samples. One assumes that (1) the $n_k$'s are independent, zero-mean Gaussian r.v.'s of variance $\sigma^2$ and (2) the $a_j$'s are uncorrelated.

The joint probability density of $K$ samples of $y$, given the set of $q_k$'s is given by:

$$p(\underline{y}|\underline{q}) = \prod_{k=1}^{K} \left[ \frac{\exp\left[ -\frac{1}{2\sigma^2} \left(y_k - a_k - \sum_{j=-\infty}^{\infty} a_j q_{k-j}\right)^2 \right]}{\sigma \sqrt{2\pi}} \right] \qquad \text{(B-2)}$$

27

The likelihood function, $L(\underline{y}|\underline{q}) \overset{\Delta}{=} \ln p\ (\underline{y}|\underline{q})$, is given by:

$$L(\underline{y}|\underline{q}) = \sum_{k=1}^{K} - \frac{1}{2\sigma^2} (y_k - a_k - \sum_{j=-\infty}^{\infty} a_j q_{k-j})^2 - \frac{K}{2} \ln(2\pi\sigma^2) \quad \text{(B-3)}$$

Our maximum likelihood estimates of $q_i$ are given by setting $\frac{\partial L}{\partial q_i} = 0$:

$$\sum_{k=1}^{K} - \frac{1}{\sigma^2} (y_k - a_k - \sum_{j=-\infty}^{\infty} a_j \hat{q}_{k-j}) a_{k-i} = 0 \quad \text{(B-4)}$$

$$\sum_{k=1}^{K} (y_k - a_k) a_{k-i} - \sum_{k=1}^{K} \sum_{j=-\infty}^{\infty} a_j \hat{q}_{k-j} a_{k-i} = 0 \quad \text{(B-5)}$$

Assuming that the $a_j$'s are uncorrelated and that the length of averaging period, K, is sufficiently long, one can assume that

$$\sum_{k=1}^{K} a_{k-j} a_{k-i} = KS\ \delta_{ij} \quad \text{(B-6)}$$

where S is the average signal power. Using equation (B-6) and defining $e_k \overset{\Delta}{=} y_k - a_k$, one gets

$$\sum_{k=1}^{K} e_k\ a_{k-i} - \sum_{j=-\infty}^{\infty} \hat{q}_j\ KS\delta_{ij} = 0 \quad \text{(B-7)}$$

28

Solving:

$$\hat{q}_i = \frac{1}{KS} \sum_{k=1}^{K} e_k a_{k-i} \qquad \qquad \text{(B-8)}$$

## APPENDIX C

## Minimization of Mean-Squared Error

Figure 4 shows a sampled data model of our system. Our goal is to minimize:

$$\overline{e_j^2} = \overline{(y_j - a_j)^2} \qquad \text{(C-1)}$$

where overbar indicates mean value. From Figure 2:

$$\overline{e_j^2} = \overline{[\sum_{v=-n}^{n} c_v \{ \sum_{w=-\infty}^{\infty} g_w a_{j-v-w} + n_{j-v} \} - a_j]^2} \qquad \text{(C-2)}$$

$$\overline{e_j^2} = \overline{[\sum_{v=-n}^{n} c_v \{ \sum_{w=-\infty}^{\infty} g_w a_{j-v-w} + n_{j-v} \} - a_j][\sum_{u=-n}^{n} c_u \{ \sum_{z=-\infty}^{\infty} g_z a_{j-u-z} + n_{j-u} \} - a_j]}$$

$$\text{(C-3)}$$

$$\overline{e_j^2} = \sum_{v=-n}^{n} \sum_{u=-n}^{n} c_v c_u [\Phi_{nn}(v-u) + \sum_{w=-\infty}^{\infty} \sum_{z=-\infty}^{\infty} g_w g_z \Phi_{aa}(v+w-u-z)]$$

$$- \sum_{v=-n}^{n} c_v \sum_{w=-\infty}^{\infty} g_w \Phi_{aa}(v+w) - \sum_{u=-n}^{n} c_u \sum_{z=-\infty}^{\infty} g_z \Phi_{aa}(u+z) + \Phi_{aa}(0)$$
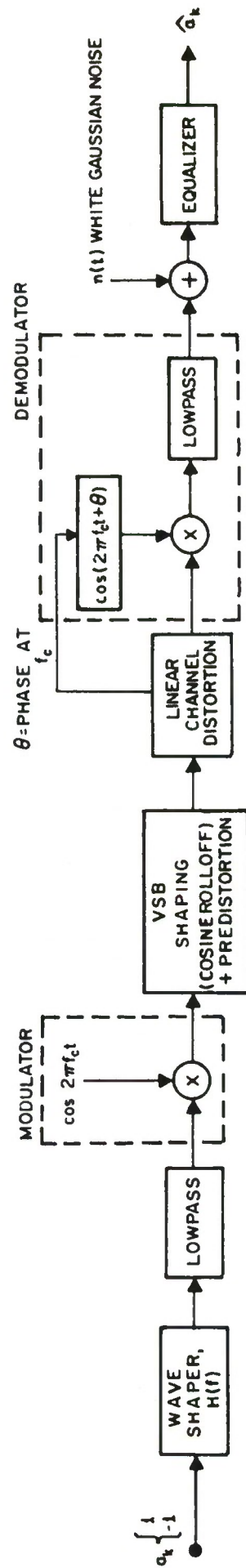
$$\text{(C-4)}$$

31

Figure 4.   SYSTEM MODEL FOR COMPUTER SIMULATION

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where

$$\phi_{nn}(u) = \overline{n(k)n(k\underline{+}u)} \qquad , \tag{C-5}$$

$$\phi_{aa}(u) = \overline{a(k)a(k\underline{+}u)} \qquad , \tag{C-6}$$

$$\phi_{na}(u) = 0 \qquad . \tag{C-7}$$

We want to minimize the mean squared error with respect to each tap:

$$\frac{\overline{\partial e_j^2}}{\partial c_k} = \sum_{u=-n}^{n} c_u[\phi_{nn}(k-u) + \sum_{w=-\infty}^{\infty} \sum_{z=-\infty}^{\infty} g_w g_z \phi_{aa}(k+w-u-z)] - \sum_{w=-\infty}^{\infty} g_w \phi_{aa}(k+w)$$

$$+ \sum_{v=-n}^{n} c_v[\phi_{nn}(v-k) + \sum_{w=-\infty}^{\infty} \sum_{z=-\infty}^{\infty} g_w g_z \phi_{aa}(v+w-k-z)] - \sum_{z=-\infty}^{\infty} g_z \phi_{aa}(k+z) \tag{C-8}$$

$$\frac{\overline{\partial e_j^2}}{\partial c_k} = 2\{ \sum_{v=-n}^{n} c_v[\phi_{nn}(v-k) + \sum_{w=-\infty}^{\infty} \sum_{z=-\infty}^{\infty} g_w g_z \phi_{aa}(v+w-z-k)] - \sum_{z=-\infty}^{\infty} g_z \phi_{aa}(z+k)\} \tag{C-9}$$

Changing indices:

$$\frac{\overline{\partial e_j^2}}{\partial c_k} = 2\{ \sum_{v=-n}^{n} [\phi_{nn}(v-k) + \sum_{z=-\infty}^{\infty} \phi_{aa}(z) \sum_{w=-\infty}^{\infty} g_w g_{w+v-z-k}]c_v - \sum_{z=-\infty}^{\infty} \phi_{aa}(z)g_{z-k}\} \tag{C-10}$$

33

Now, compare equation (C-10) with that derived from the following:

$$\overline{x_{j-k}e_j} = \left[\sum_{u=-\infty}^{\infty} g_{j-k-u}a_u + n_{j-k}\right]\left[\sum_{v=-n}^{n} c_v\left\{\sum_{w=-\infty}^{\infty} g_w a_{j-v-w} + n_{j-v}\right\} - a_j\right] \quad (C-11)$$

$$\overline{x_{j-k}e_j} = \sum_{v=-n}^{n} c_v\left[\phi_{nn}(v-k) + \sum_{u=-\infty}^{\infty}\sum_{w=-\infty}^{\infty} g_w g_{j-k-u}\phi_{aa}(v+w-j+u)\right]$$

$$- \sum_{u=-\infty}^{\infty} g_{j-k-u}\phi_{aa}(j-u) \quad (C-12)$$

Changing indices:

$$\overline{x_{j-k}e_j} = \sum_{v=-n}^{n}\left[\phi_{nn}(v-k) + \sum_{z=-\infty}^{\infty}\phi_{aa}(z)\sum_{w=-\infty}^{\infty} g_w g_{v+w-z-k}\right]c_v - \sum_{z=-\infty}^{\infty}\phi_{aa}(z)g_{z-k}$$

$$(C-13)$$

$$= \frac{1}{2}\,\overline{\frac{\partial e_j^2}{\partial c_k}} \quad (C-14)$$

34

APPENDIX D

## Convergence of the MSE Algorithm

The MSE algorithm calls for the following (equation C-13):

$$\sum_{v=-n}^{n} [\xi_{nn}(v-k) + \sum_{z=-\infty}^{\infty} \phi_{aa}(z) \sum_{w=-\infty}^{\infty} g_w g_{v+w-z-k}] c_v - \sum_{z=-\infty}^{\infty} \phi_{aa}(z) g_{z-k} = 0$$

(D-1)

where $-n \leq k \leq n$.

This can in turn be written using matrix notation:

$$(\underline{\phi}_n + \underline{G}) \underline{c} = \underline{d}$$

(D-2)

or

$$\underline{B} \underline{c} = \underline{d}$$

(D-3)

Our equalization algorithm, equation (14), becomes:

$$\underline{c}^{n+1} = \underline{c}^n - \epsilon (\underline{B} \underline{c}^n - \underline{d})$$

(D-4)

$$\underline{c}^{n+1} = (\underline{I} - \epsilon \underline{B}) \underline{c}^n - \epsilon \underline{d}$$

(D-5)

35

Convergence occurs if $\underline{I} - \epsilon \underline{B}$ has all eigenvalues within the unit circle. All eigenvalues are real because $\underline{B}$ is symmetric. Hence, the eigenvalues of $\underline{B}$ must satisfy the following:
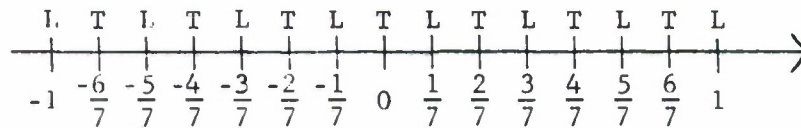
$$0 < \lambda_B < 2/\epsilon \qquad \text{(D-6)}$$

Energy considerations provide for satisfaction of the lower bound while proper choise of $\epsilon$ will gurantee that the eigenvalues are smaller than the upper bound.

APPENDIX E

Bound on Probability of Error for a Given Amount of

Distortion in an 8-Level System

For an 8-level system, the various levels and thresholds are
indicated below:

$$L \quad T \quad L \quad T \quad L \quad T \quad L \quad T \quad L \quad T \quad L \quad T \quad L \quad T \quad L$$

$$-1 \quad \frac{-6}{7} \quad \frac{-5}{7} \quad \frac{-4}{7} \quad \frac{-3}{7} \quad \frac{-2}{7} \quad \frac{-1}{7} \quad 0 \quad \frac{1}{7} \quad \frac{2}{7} \quad \frac{3}{7} \quad \frac{4}{7} \quad \frac{5}{7} \quad \frac{6}{7} \quad 1$$

Additive noise has the Gaussian density function:

$$P_N(X) = \frac{1}{\sqrt{2\pi}\ \sigma_N}\ e^{-\frac{x^2}{2\sigma_N^2}} \tag{E-1}$$

Thus, in the absence of distortion, the probability of error is
given by:

$$P_e = \frac{7}{8}\left[\int_{-\infty}^{-\frac{1}{7}} P_N(x)\ dx + \int_{\frac{1}{7}}^{\infty} P_N(x)\ dx\right] \tag{E-2}$$

where the factor of $\frac{7}{8}$ is due to the absence of thresholds above
the highest level and below the lowest level and also to the
assumption that each level is equally likely. When distortion is
present, it serves to displace the density to, in effect, give it
a non-zero mean:

37

$$P_e = \frac{7}{8} \left[ \int_{-\infty}^{-\frac{1}{7}} \frac{1}{\sqrt{2\pi}\,\sigma_N} e^{-\frac{(X-D)^2}{2\sigma_N^2}} dx + \int_{\frac{1}{7}}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_N} e^{-\frac{(X-D)^2}{2\sigma_N^2}} dx \right], \quad (E-3)$$

where D is the amount of distortion normalized to the two extreme levels. Change variables $(y = \frac{X-D}{\sigma_N})$ :

$$P_e = \frac{7}{8} \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{-1-7D}{7\sigma_N}} e^{-y^2/2} dy + \frac{1}{\sqrt{2\pi}} \int_{\frac{1-7D}{7\sigma_N}}^{\infty} e^{-y^2/2} dy \right] \qquad (E-4)$$

$$P_e = \frac{7}{8\sqrt{2\pi}} \left[ \int_{\frac{1+7D}{7\sigma_N}}^{\infty} e^{-y^2/2} dy + \int_{\frac{1-7D}{7\sigma_N}}^{\infty} e^{-y^2/2} dy \right] \qquad (E-5)$$

$$P_e = \frac{7}{8} \left[ \mathrm{erfc}_* \left( \frac{1+7D}{7\sigma_N} \right) + \mathrm{erfc}_* \left( \frac{1-7D}{7\sigma_N} \right) \right] \qquad (E-6)$$

where $\mathrm{erfc}_*$ (:) is defined on page 37 of Van Trees.[14]

38

Figure 5 is a plot of $p_e$ vs. D for various S/N ratios that demonstrates how harmful distortion can be.

For a 4-level system, the probability of error, given by an equality similar to equation (E-6) is

$$p_e = \frac{3}{4} \left[ erfc_* \left(\frac{1+3D}{3\sigma_N}\right) + erfc_*\left(\frac{1-3D}{3\sigma_N}\right) \right] \quad , \qquad (E-7)$$

where D is the distortion normalized to the extreme levels ($\pm 1$) - if D = 1, the binary eye is closed. Figure 6 gives the $p_e$ vs. D plots for 4-level transmission.
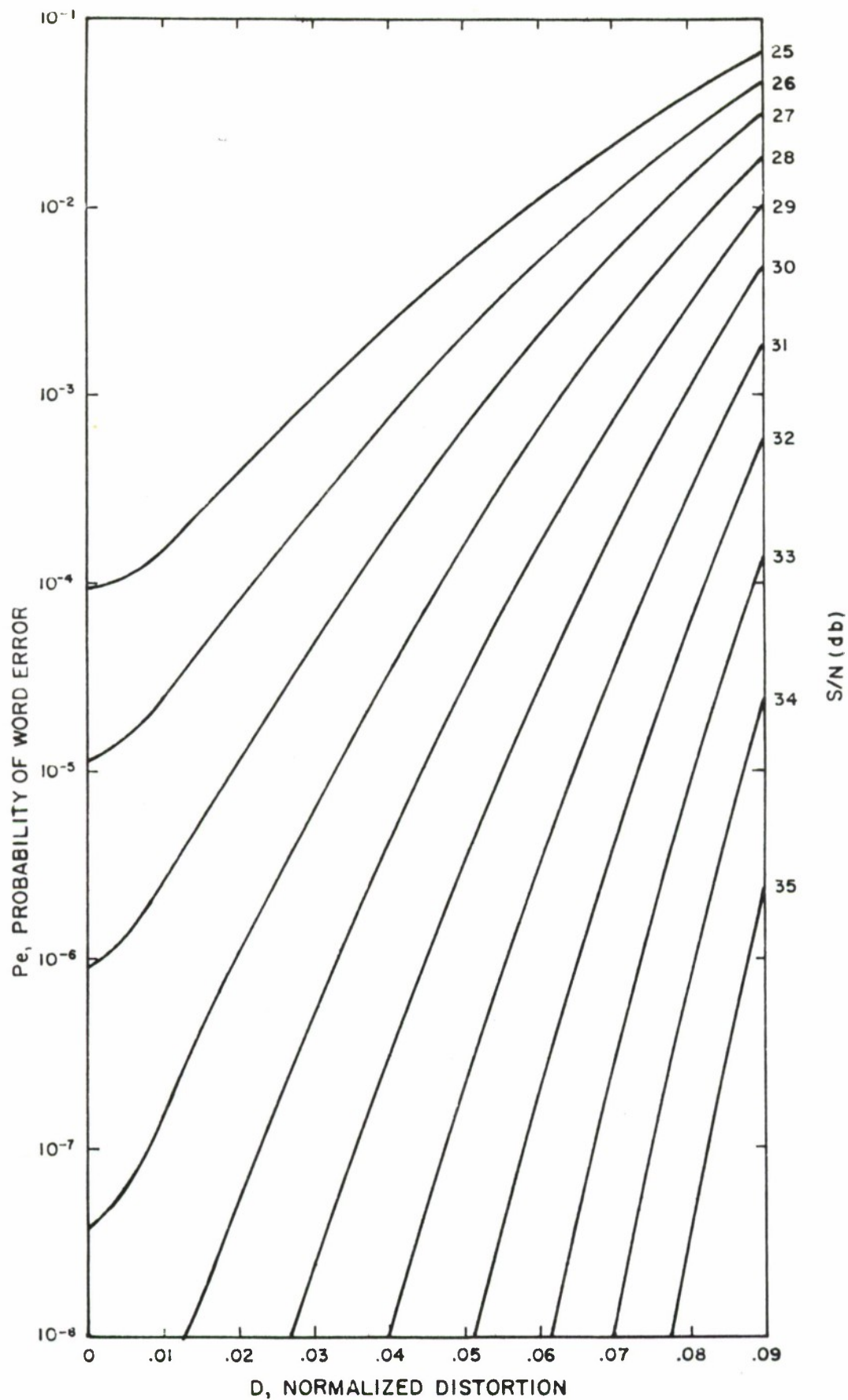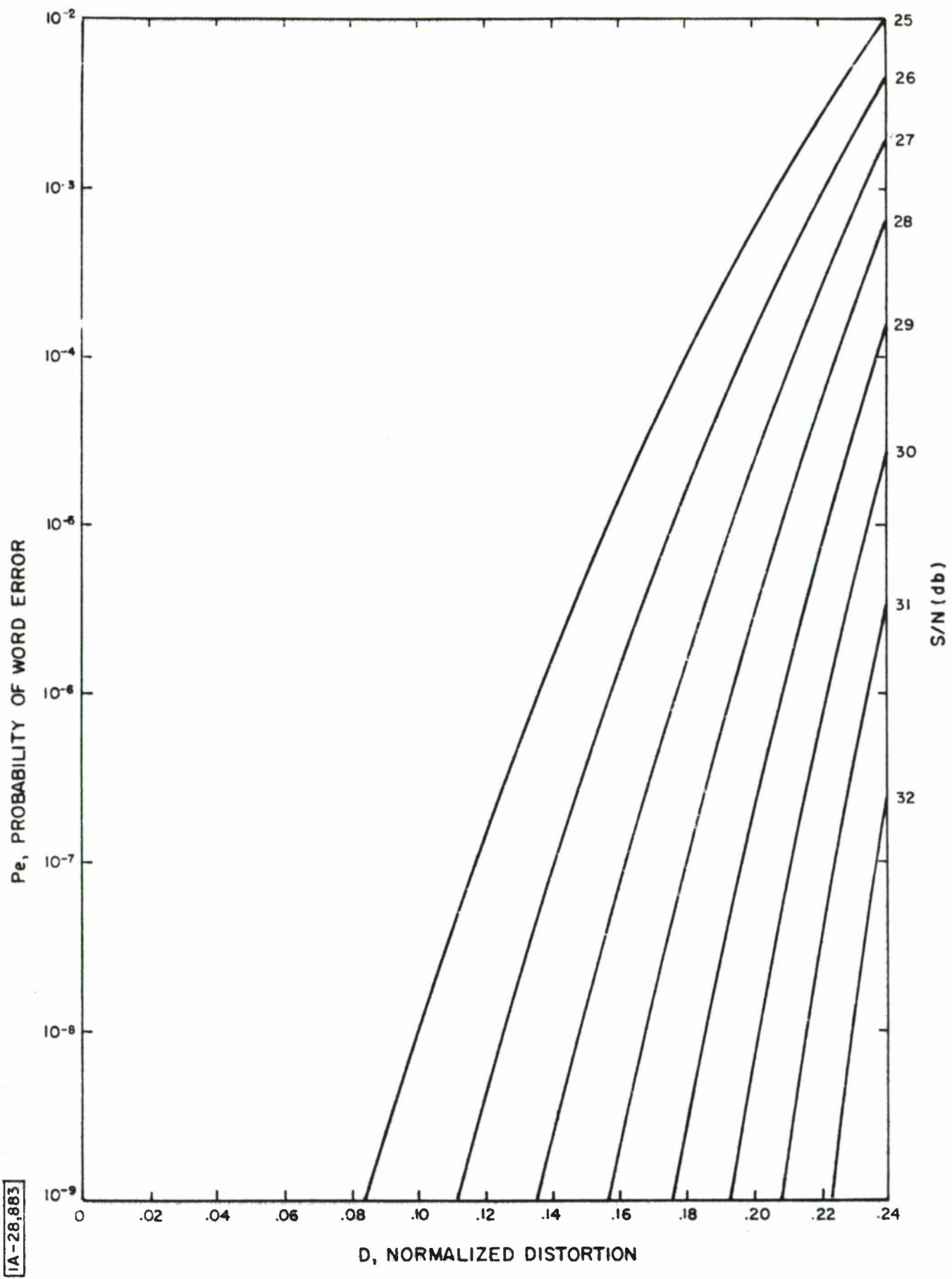
39

Figure 5. Pe vs. D FOR AN 8-LEVEL SYSTEM

Figure 6. Pe vs. D FOR A 4-LEVEL SYSTEM

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**13. ABSTRACT**

This paper is a revision of WP-2888. It reviews the present state-of-the-art techniques of narrowband (4 KHz) wireline equalization for PAM signals. Two basic minimization techniques are discussed. Related derivations and proofs are included. In addition to linear distortion, the problems due to phase jitter are mentioned as areas of possible future research.

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| 14. KEY WORDS | LINK A | | LINK B | | LINK C | |
|---|---|---|---|---|---|---|
| | ROLE | WT | ROLE | WT | ROLE | WT |
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| TRANSVERSAL EQUALIZER | | | | | | |
| TRANSVERSAL EQUALIZATION | | | | | | |
| ADAPTIVE DIGITAL FILTERING | | | | | | |
| MULTI-LEVEL PAM | | | | | | |